# High Dimensional Expansion

Roy Meshulam
Technion – Israel Institute of Technology

Hausdorff Research Institute for Mathematics
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# Plan

## Coboundary Expansion

- The $k$-dimensional Cheeger constant
- Homology of random complexes
- Expansion and topological overlap
- Expansion via symmetry
- 2-expanders from random Latin squares

## Spectral Expansion

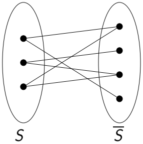
- Spectral gap of the $k$-Laplacian
- Spectral gap and colored simplices
- Garland's method
- Homology of random flag complexes
- Spectral gap and hypergraph matching

# Graphical Cheeger Constant

## Edge Cuts

For a graph $G = (V, E)$ and $S \subset V$, $\overline{S} = V - S$ let

$$e(S, \overline{S}) = |\{e \in E : |e \cap S| = 1\}|.$$




## Cheeger Constant

$$h(G) = \min_{0<|S|\leq \frac{|V|}{2}} \frac{e(S, \overline{S})}{|S|}.$$

# Graphical Spectral Gap

## Laplacian Matrix

$G = (V, E)$ a graph, $|V| = n$.

The Laplacian of $G$ is the $V \times V$ matrix $L_G$:

$$L_G(u,v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise.} \end{cases}$$

## Eigenvalues of $L_G$

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).$$

$$\lambda_2(G) = \text{Spectral Gap of } G.$$

# Cheeger Constant vs. Spectral Gap

Theorem [Alon-Milman, Tanner]:

For all $\emptyset \neq S \subsetneq V$

$$e(S, \overline{S}) \geq \frac{|S||\overline{S}|}{n} \lambda_2(G).$$

In particular

$$h(G) \geq \frac{\lambda_2(G)}{2}.$$

Theorem [Alon, Dodziuk]:

If $G$ is $d$-regular then

$$h(G) \leq \sqrt{2d\lambda_2(G)}.$$

$h(G)$ and $\lambda_2(G)$ are therefore essentially equivalent measures of graphical expansion.

# High Dimensional Expansion

The notions of Cheeger Constant and Spectral Gap have natural high dimensional extensions. They are however not equivalent in dimensions greater than one.

Coboundary Expansion

- Linial-M-Wallach: Homology of random complexes.
- Gromov: The topological overlap property.
- Gundert-Wagner: Expansion of random complexes.

Spectral Expansion

- Garland: Cohomology of discrete groups.
- Aharoni-Berger-M: Hypergraph matching.
- Kahle: Homology of random flag complexes.

# Simplicial Cohomology

$X$ a simplicial complex on $V$, $R$ a fixed abelian group.
$i$-face of $\sigma = [v_0, \cdots, v_k]$ is $\sigma_i = [v_0, \cdots, \widehat{v_i}, \cdots, v_k]$.
$C^k(X) =$ $k$-cochains $=$ skew-symmetric maps $\phi : X(k) \to R$.
Coboundary Operator $d_k : C^k(X) \to C^{k+1}(X)$ given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

$d_{-1} : C^{-1}(X) = R \to C^0(X)$ given by
$d_{-1}a(v) = a$ for $a \in R$ , $v \in V$.
$Z^k(X) =$ $k$-cocycles $= \ker(d_k)$.
$B^k(X) =$ $k$-coboundaries $= \mathrm{Im}(d_{k-1})$.
$k$-th reduced cohomology group of $X$:

$$\tilde{\mathrm{H}}^k(X) = \tilde{\mathrm{H}}^k(X; R) = Z^k(X)/B^k(X) \quad .$$

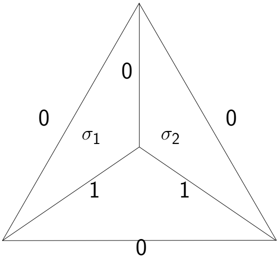
# Cut of a Cochain

Cut determined by a $k$-cochain $\phi \in C^k(X; R)$:

$$\mathrm{supp}(d_k\phi) = \{\tau \in X(k+1) \ : \ d_k\phi(\tau) \neq 0\} \ .$$

Cut Size of $\phi$: $\|d_k\phi\| = |\mathrm{supp}(d_k\phi)|$.

Example:



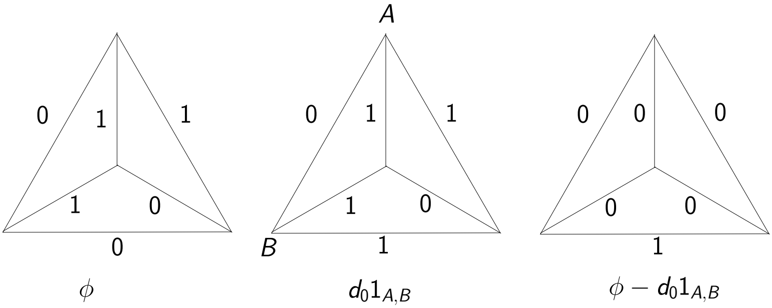

$$\|d_1\phi\| = |\{\sigma_1, \sigma_2\}| = 2$$

# Cosystolic Norm of a Cochain

The Cosystolic Norm of a $k$-cochain $\phi \in C^k(X;R)$:

$$\|[\phi]\| = \min \ \{ \ |\mathrm{supp}(\phi + d_{k-1}\psi)| \ : \ \psi \in C^{k-1}(X;R) \ \}.$$

Example: $\|\phi\| = 3$ but $\|[\phi]\| = 1$



# Expansion of a Complex

## Expansion of a Cochain

The expansion of $\phi \in C^k(X;R) - B^k(X;R)$ is

$$\frac{\|d_k\phi\|}{\|[\phi]\|}.$$

## $k$-expansion Constant

$$h_k(X;R) = \min\left\{\frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X;R) - B^k(X;R)\right\}.$$

## Remarks:

- $G$ graph $\quad\Rightarrow\quad h_0(G;\mathbb{F}_2) = h(G)$.
- $h_k(X;R) > 0 \quad\Leftrightarrow\quad \tilde{H}^k(X;R) = 0$.
- In the sequel: $h_k(X) = h_k(X;\mathbb{F}_2)$.

# Expansion of a Simplex

$\Delta_{n-1}$ = the $(n-1)$-dimensional simplex on $V = [n]$.

Claim [M-Wallach, Gromov]:

$$h_{k-1}(\Delta_{n-1}) \geq \frac{n}{k+1}.$$

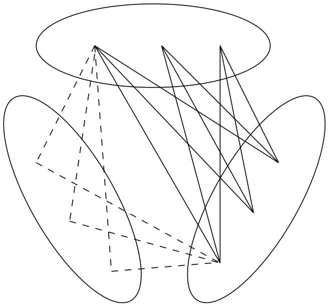
Example:

$[n] = \bigcup_{i=0}^{k} V_i \ , \ |V_i| = \frac{n}{k+1}$

$\phi = 1_{V_0 \times \cdots \times V_{k-1}}$

$\|[\phi]\| = (\frac{n}{k+1})^k$

$\|d_{k-1}\phi\| = (\frac{n}{k+1})^{k+1}$



# A Model of Random Complexes

$Y$ a simplicial complex , $Y^{(i)} = i$-dim skeleton of $Y$.
$Y(i) =$ oriented $i$-dim simplices of $Y$.
$f_i(Y) = |Y(i)|$.
$\Delta_{n-1} =$ the $(n-1)$-dimensional simplex on $V = [n]$.

$Y_k(n,p) =$ probability space of all complexes

$$\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$$

with probability distribution

$$\Pr(Y) = p^{f_k(Y)}(1-p)^{\binom{n}{k+1} - f_k(Y)} .$$

# Homological Connectivity of Random Complexes

Fix $k \geq 1$ and a finite abelian group $R$.

Theorem [Linial-M '03 , M-Wallach '06]:

For any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr\left[\, Y \in Y_k(n,p) : \tilde{\mathrm{H}}_{k-1}(Y;R) = 0 \,\right] = \begin{cases} 0 & p = \frac{k\log n - \omega(n)}{n} \\ 1 & p = \frac{k\log n + \omega(n)}{n} \end{cases} .$$

The Relevance of Expansion:

If $0 \neq [\phi] \in \tilde{\mathrm{H}}^{k-1}(\Delta_{n-1}^{(k-1)})$ then

$$\Pr\ \left[\, [\phi] \in \tilde{\mathrm{H}}^{k-1}(Y;\mathbb{F}_2) \,\right] = (1-p)^{\|d_k\phi\|}$$
$$\leq (1-p)^{\frac{n\|[\phi]\|}{k+1}} .$$

# Weighted Expansion

$X$ - $n$-dimensional pure simplicial complex.
A probability distribution on $X(k)$:

$$w(\sigma) = \frac{|\{\eta \in X(n) : \sigma \subset \eta\}|}{\binom{n+1}{k+1} f_n(X)}.$$

For $\phi \in C^k(X)$ let

$$\|\phi\|_w = \sum_{\{\sigma \in X(k) : \phi(\sigma) \neq 0\}} w(\sigma)$$

$$\|[\phi]\|_w = \min\{\|\phi + d_{k-1}\psi\|_w : \psi \in C^{k-1}(X)\}.$$

Weighted $k$-th Expansion:

$$\underline{h}_k(X) = \min\left\{\frac{\|d_k\phi\|_w}{\|[\phi]\|_w} : \phi \in C^k(X) - B^k(X)\right\}.$$

# The Affine Overlap Property

## Number of Intersecting Simplices

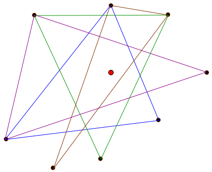
For $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_A(p) = |\{\sigma \subset [n] : |\sigma| = k+1 \ , \ p \in \mathrm{conv}\{a_i\}_{i\in\sigma}\}|.$$

## Theorem [Bárány]:

There exists $p \in \mathbb{R}^k$ such that

$$f_A(p) \geq \frac{1}{(k+1)^k}\binom{n}{k+1} - O(n^k).$$



# The Topological Overlap Property

## Number of Intersecting Images

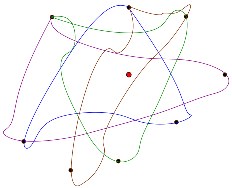
For a continuous map $f : \Delta_{n-1} \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in \Delta_{n-1}(k) : p \in f(\sigma)\}|.$$

## Theorem [Gromov]:

There exists $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \frac{2k}{(k+1)!(k+1)} \binom{n}{k+1} - O(n^k).$$



# Topological Overlap and Expansion

## Number of Intersecting Images

For a continuous map $f : X \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in X(k) : p \in f(\sigma)\}|.$$

## Expansion Condition on $X$

Suppose that for all $0 \leq i \leq k-1$

$$\underline{h}_i(X) \geq \epsilon.$$

## Theorem [Gromov]

There exists a $\delta = \delta(k, \epsilon)$ such that for any continuous map $f : X \to \mathbb{R}^k$ there exists a $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \delta f_k(X).$$

# Example: Symmetric Matroids

Matroid:

An $n$-dimensional simplicial complex $M \subset 2^V$ such that $M[S]$ is pure for all $S \subset V$.

Example: Linear Matroids

$A$ - finite subset of a vector space.
$M =$ all linearly independent subsets $\sigma \subset A$.

Homology of matroids:

$\tilde{H}_i(M) = 0$ for all $0 \leq i \leq \dim M - 1$.

Symmetric matroid:

$G = Aut(M)$ is transitive on the maximal faces.

# Some Symmetric Matroids

## The Partition Matroid $X_{n,m}$

Let $V_1, \ldots, V_{n+1}$ be $n+1$ disjoint sets, $|V_i| = m$.

$$X_{n,m} = \{\sigma \subset \bigcup_{i=1}^{n+1} V_i : \forall i \;\; |\sigma \cap V_i| \leq 1\}.$$

## Independence Matroid of Affine Space

$$\mathrm{IN}(\mathbb{F}_q^n) = \{\sigma \subset \mathbb{F}_q^n : \sigma \textit{ is linearly independent}\}.$$

## Hermitian Unital with 65 Points

Independence matroid of the curve

$$H = \{[x, y, z] \in PG(2, 16) : x^5 + y^5 + z^5 = 0\}.$$

# Expansion of Symmetric Matroids

Proposition [Lubotzky-M-Mozes]:

$M$ symmetric matroid $\Rightarrow$ $\underline{h}_k(M) \geq 8^{-\dim M}$ $\forall\, k \leq \dim M - 1$.

Example: The Partition Matroid $X_{n,m}$

For $0 \leq k \leq n-1$

$$\underline{h}_k(X_{n,m}) \geq \frac{\binom{n+1}{k+1}}{\sum_{j=0}^{k+1} \left(\frac{2(m-1)}{m}\right)^j \binom{n-j}{n-k-1}}.$$

In particular

$$\underline{h}_k(\text{Octahedral } n-\text{sphere}) = \underline{h}_k(X_{n,2}) \geq 1$$

and

$$\underline{h}_{n-1}(X_{n,m}) \geq \frac{n+1}{\sum_{j=0}^{n} \left(\frac{2(m-1)}{m}\right)^j} > \frac{n+1}{2^{n+1}-1}.$$

# Example: The Spherical Buildings $\Delta = A_{n+1}(\mathbb{F}_q)$

Vertices: All nontrivial linear subspaces $0 \neq V \subsetneq \mathbb{F}_q^{n+2}$.
Simplices: $V_0 \subset \cdots \subset V_k$.

Homology of $\Delta$ [Solomon, Tits]:
$\tilde{\mathrm{H}}_i(\Delta) = 0$ for $i < n$ and $\dim \tilde{\mathrm{H}}_n(\Delta) = q^{\binom{n+2}{2}}$.

Proposition [Gromov, LMM]:

$$\underline{h}_{n-1}(A_{n+1}(\mathbb{F}_q)) \geq \frac{1}{(n+2)!}.$$

Problem:
For fixed $n \geq 2$ determine

$$\lim_{q\to\infty} \underline{h}_{n-1}(A_{n+1}(\mathbb{F}_q)).$$

# Expander Graphs

## $(d, \epsilon)$-Expanders

A family of graphs $\{G_n = (V_n, E_n)\}_n$ with $|V_n| \to \infty$ with two seemingly contradicting properties:

- High Connectivity: $h(G_n) \geq \epsilon$.
- Sparsity: $\max_v \deg_{G_n}(v) \leq d$.

Pinsker:

Random $3 \leq d$-regular graphs are $(d, \epsilon)$-expanders.

Margulis:

Explicit construction of expanders.

Lubotzky-Phillips-Sarnak, Margulis:

Ramanujan Graphs - an "optimal" family of expanders.

# Expander Complexes

## Degree of a Simplex

For $\sigma \in X(k-1)$ let $\deg(\sigma) = |\{\tau \in X(k) : \sigma \subset \tau\}|$.
$D_{k-1}(X) = \max_{\sigma \in X(k-1)} \deg(\sigma)$.

## $(k, d, \epsilon)$-Expanders

A family of Complexes $\{X_n\}_n$ with $f_0(X_n) \to \infty$ such that

$$D_{k-1}(X_n) \leq d \quad \text{and} \quad h_{k-1}(X_n) \geq \epsilon.$$

## Problems

For fixed fixed $k \geq 2, d, \epsilon > 0$ construct:

- $(k, d, \epsilon)$-expanders.
- Complexes that are jointly $(j, d, \epsilon)$-expanders for all $j \leq k$.

# Latin Squares

## Definitions

$\mathbb{S}_n$ = Symmetric group on $[n]$.
$(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is legal if $\pi_i(\ell) \neq \pi_j(\ell)$ for all $\ell$ and $i \neq j$.
A Latin Square is a legal $n$-tuple $L = (\pi_1, \ldots, \pi_n) \in \mathbb{S}_n^n$.
$\mathcal{L}_n$ = Latin squares of order $n$ with uniform measure.

## The Usual Picture

$L = (\pi_1, \ldots, \pi_n) \leftrightarrow T_L \in M_{n \times n}([n])$
$T_L(i, \pi_k(i)) = k$ for $1 \leq i, k \leq n$.

## Example for $n = 4$

$\pi = (1234)$

$L = (Id, \pi, \pi^2, \pi^3)$

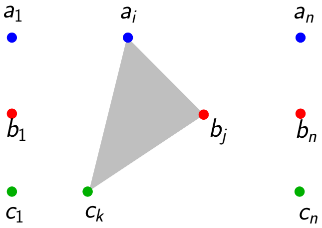
$T_L =$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 4 | 1 | 2 | 3 |
| 3 | 4 | 1 | 2 |
| 2 | 3 | 4 | 1 |

# The Complete 3-Partite Complex

$$V_1 = \{a_1, \ldots, a_n\}\ ,\ V_2 = \{b_1, \ldots, b_n\}\ ,\ V_3 = \{c_1, \ldots, c_n\}$$

$$T_n = V_1 * V_2 * V_3 = \{\sigma \subset V : |\sigma \cap V_i| \leq 1 \text{ for } 1 \leq i \leq 3\}$$




$$T_n \simeq S^2 \vee \cdots \vee S^2 \qquad (n-1)^3 \text{ times}$$

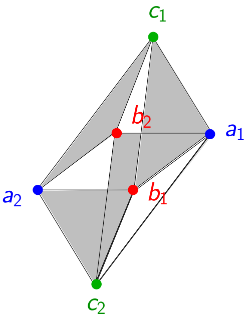
# Latin Square Complexes

$L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$ defines a complex $Y(L) \subset T_n$ by

$$Y(L)(2) = \left\{ \, [a_i, b_j, c_{\pi_i(j)}] \quad : \quad 1 \le i, j \le n \right\}.$$

Example: $n = 2$

$$L = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \qquad Y(L) =$$




$$Y\left( \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \right) \cup Y\left( \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \right) = T_2$$

# Random Latin Squares Complexes

## Multiple Latin Squares

For $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$ let $Y(\underline{L}^d) = \cup_{i=1}^d Y(L_i)$.

## The Probability Space $\mathcal{Y}(n, d)$

$\mathcal{L}_n^d = d$-tuples of Latin squares of order $n$ with uniform measure.
$\mathcal{Y}(n, d) = \{Y(\underline{L}^d) : \underline{L}^d \in \mathcal{L}_n^d\}$ with induced measure from $\mathcal{L}_n^d$.

## Theorem [Lubotzky-M]:

There exist $\epsilon > 0, d < \infty$ such that

$$\lim_{n\to\infty} \Pr\left[Y \in \mathcal{Y}(n, d) : h_1(Y) > \epsilon\right] = 1.$$

Remark: $\epsilon = 10^{-11}$ and $d = 10^{11}$ will do.

# Idea of Proof

Fix $0 < c < 1$ and let $\phi \in C^1(T_n; \mathbb{F}_2)$.

$$\phi \text{ is } \begin{cases} c-\text{small} & \text{if } \|[\phi]\| \leq cn^2 \\ c-\text{large} & \text{if } \|[\phi]\| \geq cn^2 \end{cases}$$

## $c$-Small Cochains

Lower bound on expansion in terms of the spectral gap of the vertex links.

## $c$-Large Cochains

Expansion is obtained by means of a new large deviations bound for the probability space $\mathcal{L}_n$ of Latin squares.

# 2-Expansion and Spectral Gap

### Notation

For a complex $T_n^{(1)} \subset Y \subset T_n$ let:
$Y_v = \mathrm{lk}(Y, v) =$ the link of $v \in V$.
$\lambda_v =$ spectral gap of the $n \times n$ bipartite graph $Y_v$.
$\tilde{\lambda} = \min_{v \in V} \lambda_v$.
$d = D_1(Y) =$ maximum edge degree in $Y$.

### Theorem [LM]:

If $\|[\phi]\| \leq cn^2$ then

$$\|d_1\phi\| \geq \left( \frac{(1 - c^{1/3})\tilde{\lambda}}{2} - \frac{d}{3} \right) \|[\phi]\|.$$

# Large Deviations for Latin Squares

## The Random Variable $f_{\mathcal{E}}$

$\mathcal{E}$ - a family of 2-simplices of $T_n$, $|\mathcal{E}| \geq cn^3$.
For a Latin square $L \in \mathcal{L}_n$ let

$$f_{\mathcal{E}}(L) = |Y(L) \cap \mathcal{E}|.$$

Then

$$E[f_{\mathcal{E}}] = \frac{|\mathcal{E}|}{n} \geq cn^2.$$

## Theorem [LM]:

For all $n \geq n_0(c)$

$$\Pr[L \in \mathcal{L}_n \; : \; f_{\mathcal{E}}(L) < 10^{-3} c^2 n^2] < e^{-10^{-3} c^2 n^2}.$$

# Higher Laplacians

A positive weight function $c(\sigma)$ on the simplices of $X$ induces an Inner product on $C^k(X) = C^k(X;\mathbb{R})$:

$$(\phi, \psi) = \sum_{\sigma \in X(k)} c(\sigma)\phi(\sigma)\psi(\sigma) \quad .$$

Adjoint $\quad d_k^* : C^{k+1}(X) \to C^k(X)$

$$(d_k\phi, \psi) = (\phi, d_k^*\psi) \ .$$

$$C^{k-1}(X) \underset{d_{k-1}^*}{\overset{d_{k-1}}{\rightleftarrows}} C^k(X) \underset{d_k^*}{\overset{d_k}{\rightleftarrows}} C^{k+1}(X)$$

The reduced $k$-Laplacian of $X$ is the positive semidefinite operator

$$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X) \quad .$$

# Matrix Representation of $\Delta_k$

For the constant weight function $c \equiv 1$, the matrix form of the Laplacian is

$$\Delta_k(\sigma,\tau) = \begin{cases} \deg(\sigma) + k + 1 & \sigma = \tau \\ (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & |\sigma \cap \tau| = k \ , \ \sigma \cup \tau \notin X \end{cases}$$

## Relation with the Graph Laplacian

Let $G \ = \ $ 1-skeleton of $X$

$$\Delta_0 = L_G + J$$

$$\mu_0(X) = \lambda_2(G)$$

# Harmonic Cochains

The space of Harmonic $k$-cochains

$$\ker \Delta_k = \{\phi \in C^k(X) : d_k\phi = 0 \ , \ d_{k-1}^*\phi = 0\}.$$

## Simplicial Hodge Theorem:

$$C^k(X) = \mathrm{Im}\, d_{k-1} \oplus \ker \Delta_k \oplus \mathrm{Im}\, d_k^* \ .$$

$$\ker \Delta_k \cong \tilde{\mathrm{H}}^k(X;\mathbb{R}).$$

$\mu_k(X) =$ minimal eigenvalue of $\Delta_k$.

## A Vanishing Criterion:

$$\mu_k(X) > 0 \Leftrightarrow \tilde{\mathrm{H}}_k(X;\mathbb{R}) = 0.$$

# Spectral Gap and Colorful Simplices

$\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ with vertex coloring: $[n] = V_0 \cup \cdots \cup V_k$.
Number of colorful $k$-simplices:

$$e(V_0, \ldots, V_k) = |\{\sigma \in X(k) : |\sigma \cap V_i| = 1 \ \ \forall\, 0 \leq i \leq k\}|.$$

Theorem [Parzanchevski-Rosenthal-Tessler]:

Let $c$ be the constant weight function $c(\sigma) \equiv 1$. Then

$$e(V_0, \ldots, V_k) \geq \frac{\prod_{i=0}^{k} |V_i|}{n} \cdot \mu_{k-1}(X).$$

# Sketch of Proof

Define $\psi \in C^k(\Delta_{n-1})$ by

$$\psi([v_0, \ldots, v_k]) = \begin{cases} sgn(\pi) & v_{\pi(i)} \in V_i \ \ \forall\, 0 \leq i \leq k \\ 0 & [v_0, \ldots, v_k] \text{ is not colorful.} \end{cases}$$

Let $\phi = d_{k-1}^* \psi \in C^{k-1}(\Delta_{n-1}) = C^{k-1}(X)$. Then:

$$(\Delta_{k-1}\phi, \phi) = (d_{k-1}\phi, d_{k-1}\phi) = n^2 \cdot e(V_0, \ldots, V_k)$$

$$(\phi, \phi) = n \prod_{i=0}^{k} |V_i|.$$

Therefore, by the variational principle:

$$\mu_{k-1}(X) \leq \frac{(\Delta_{k-1}\phi, \phi)}{(\phi, \phi)} = \frac{n \cdot e(V_0, \ldots, V_k)}{\prod_{i=0}^{k} |V_i|}.$$

# Eigenvalues and Cohomology

Let $X$ be a pure $d$-dimensional complex with weight function:

$$c(\sigma) = (d - \dim \sigma)! |\{\tau \in X(d) : \tau \supset \sigma\}|.$$

For $\tau \in X$ consider the link $X_\tau = \mathrm{lk}(X, \tau)$ with a weight function given by $c_\tau(\alpha) = c(\tau\alpha)$.

Theorem [Garland '72]:

Let $0 \leq \ell < k < d$. Then:

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell + 1}{k + 1} \quad \Rightarrow \quad H^k(X; \mathbb{R}) = 0.$$

In particular:

$$\min_{\tau \in X(d-2)} \mu_0(X_\tau) > \frac{d - 1}{d} \quad \Rightarrow \quad H^{d-1}(X; \mathbb{R}) = 0.$$

# Complexes with Expanding Links

## The Projective Plane Graph

$G_q = (V_q, E_q)$ : points vs. lines graph of $PG(2, q)$.

$$|V_q| = 2(q^2 + q + 1) \quad , \quad |E_q| = (q+1)(q^2 + q + 1).$$

Spectral Gap: $\mu_0(G_q) = 1 - \frac{\sqrt{q}}{q+1}$.

If $q \geq d^2$ then $\mu_0(G_q) > \frac{d-1}{d}$. This implies the following

**Theorem [Garland]:**

Let $q \geq d^2$ and let $X$ be a pure $d$-dimensional complex such that $\mathrm{lk}(X, \tau) \cong G_q$ for all $\tau \in X(d-2)$.
Then $H_{d-1}(X; \mathbb{R}) = 0$.

# Cohomology of Discrete Subgroups

$\mathbb{K}$ a local field with residue field $\mathbb{F}_q$.
$\Gamma$ a torsion-free discrete cocompact subgroup of $SL_{d+1}(\mathbb{K})$.

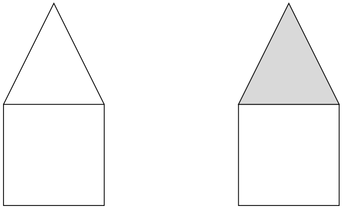
Theorem [Garland]:
If $q \geq d^2$ then $H^i(\Gamma; \mathbb{R}) = 0$ for $0 < i < d$.

Sketch of Proof:
$\mathcal{B} = \tilde{A}_d(\mathbb{K})$ - the affine building associated to $SL_{d+1}(\mathbb{K})$.
$\mathcal{B}$ is a contractible complex with a free $\Gamma$ action.
The quotient space $\mathrm{B}\Gamma = \mathcal{B}/\Gamma$ is a pure $d$-dimensional complex such that $\mathrm{lk}(\mathrm{B}\Gamma, \tau) \cong G_q$ for all $\tau \in \mathrm{B}\Gamma(d-2)$.
Therefore for all $0 < i < d$

$$H^i(\Gamma; \mathbb{R}) = H^i(\mathrm{B}\Gamma; \mathbb{R}) = 0.$$

# Flag Complexes

The flag complex $X(G)$ of a graph $G = (V, E)$:
Vertex set: $V$, Simplices: all cliques $\sigma$ of $G$.



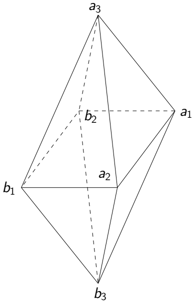
$G$ $X(G)$

Remark:
The first subdivision of a complex is a flag complex.

# Face Numbers of Flag Complexes

## Octahedral $n$-Sphere

$$(S^0)^{*(k+1)} = \{a_1, b_1\} * \cdots * \{a_{k+1}, b_{k+1}\}$$




**Proposition [M '03]:**
If $\tilde{\mathrm{H}}_k(X(G)) \neq 0$ then for all $j$:

$$f_j(X(G)) \geq f_j((S^0)^{*(k+1)}) = \binom{k+1}{j+1} 2^{j+1}.$$

# Homology of Flag Complexes of Random Graphs

Let $\epsilon > 0$ be fixed and let $G \in G(n, p)$.

Theorem [Kahle '12]:

$$p \leq n^{-\frac{1}{k}-\epsilon} \quad \Rightarrow \quad H_k(X(G); \mathbb{Z}) = 0 \ \text{ a.a.s.}$$

$$p \geq \left( \frac{(\frac{k}{2} + 1 + \epsilon) \log n}{n} \right)^{\frac{1}{k+1}} \quad \Rightarrow \quad H_k(X(G); \mathbb{R}) = 0 \ \text{ a.a.s.}$$

Theorem [DeMarco-Hamm-Kahn '12]:

$$p \geq \left( \frac{(\frac{3}{2} + \epsilon) \log n}{n} \right)^{\frac{1}{2}} \quad \Rightarrow \quad H_1(X(G); \mathbb{F}_2) = 0 \ \text{ a.a.s.}$$

# Eigenvalues of Flag Complexes

$G = (V, E)$ graph, $|V| = n$, $X = X(G)$ with weights $c(\sigma) \equiv 1$.
$\mu_k = \mu_k(X) =$ minimal eigenvalue of $\Delta_k$ on $X$.

Theorem [Aharoni-Berger-M]:

For $k \geq 1$

$$k\mu_k \geq (k+1)\mu_{k-1} - n.$$

In particular:

$$\mu_k \geq (k+1)\lambda_2 - kn.$$

Corollary:

$$\lambda_2(G) > \frac{kn}{k+1} \Rightarrow \mu_k > 0 \Rightarrow \tilde{\mathrm{H}}^k(X(G)) = 0.$$

# Example: Turán Graph

$|V_1| = \cdots = |V_k| = \ell$, $n = k\ell$, $m = (\ell - 1)^k$.
$T_k(n)$ - the complete $k$-partite graph on $V_1 \cup \cdots \cup V_k$.

## Spectral gap

$$\lambda_2(T_k(n)) = \frac{(k-1)n}{k}.$$

## Flag complex

$$X(T_k(n)) = V_1 * \cdots * V_k \simeq \bigvee_{i=1}^{m} S^{k-1}.$$

$$\dim \tilde{H}_{k-1}(X(T_k(n)); \mathbb{R}) = m \neq 0.$$

# Eigenvalues and Connectivity of $\mathrm{I}(G)$

## The independence complex $\mathrm{I}(G)$

Vertex set: $V$, Simplices: all independent sets $\sigma$ of $G$.

## Homological connectivity

$$\eta(Y) = 1 + \min\{i : \tilde{H}_i(Y) \neq 0\}.$$

## Theorem [ABM]:

For a graph $G$ on $n$ vertices

$$\eta(\mathrm{I}(G)) \geq \frac{n}{\lambda_n(G)}.$$

# Bipartite Matching

$A_1, \ldots, A_m$ finite sets.
A System of Distinct Representatives (SDR):
a choice of distinct $x_1 \in A_1, \ldots, x_m \in A_m$.

| $A_1$ | $A_2$ | $A_3$ |
|---|---|---|
| 1 | 1 | |
| | | 2 |
| 3 | 3 | 3 |

$\exists$ SDR

| $A_1$ | $A_2$ | $A_3$ |
|---|---|---|
| 1 | 1 | |
| 2 | | 2 |

$\not\exists$ SDR

Hall's Theorem (1935)

$(A_1, \ldots, A_m)$ has an SDR iff
$|\cup_{i \in \mathrm{I}} A_i| \geq |\mathrm{I}|$ for all $\mathrm{I} \subset [m] = \{1, \ldots, m\}$.

# Hypergraph Matching

A Hypergraph is a family of sets $\mathcal{F} \subset 2^V$
$(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ a sequence of $m$ hypergraphs
A System of Disjoint Representatives (SDR) for $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ is a choice of pairwise disjoint $F_1 \in \mathcal{F}_1, \ldots, F_m \in \mathcal{F}_m$

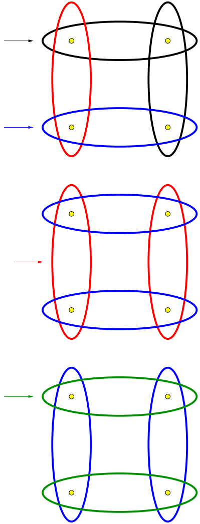
When do $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ have an SDR?

The problem is NP-Complete even if all $\mathcal{F}_i$'s consist of 2-element sets. Therefore, we cannot expect a "good" characterization as in Hall's Theorem.

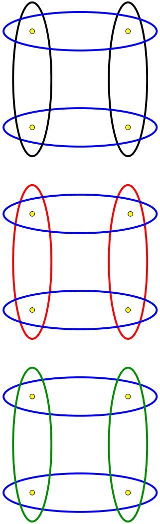
There are however some interesting sufficient conditions ...

# Do $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ have an SDR?

$\exists$ SDR



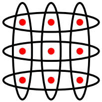
$\nexists$ SDR



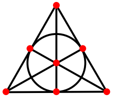
# The Aharoni-Haxell Theorem

A Matching is a hypergraph $\mathcal{M}$ of pairwise disjoint sets.
The Matching Number $\nu(\mathcal{F})$ of a hypergraph $\mathcal{F}$ is the maximal size $|\mathcal{M}|$ of a matching $\mathcal{M} \subset \mathcal{F}$.

$\nu(\mathcal{F}) = 3$



$\nu(\mathcal{F}) = 1$



The Aharoni-Haxell Theorem
$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $\mathrm{I} \subset [m]$

$$\nu\left(\bigcup_{i \in \mathrm{I}} \mathcal{F}_i\right) > r(|\mathrm{I}| - 1) \quad .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.

# A Fractional Extension

A Fractional Matching of a hypergraph $\mathcal{F}$ on $V$ is a function $f : \mathcal{F} \rightarrow \mathbb{R}_+$ such that $\sum_{F \ni v} f(F) \leq 1$ for all $v \in V$.
The Fractional Matching Number $\nu^*(\mathcal{F})$ is $\max_f \sum_{F \in \mathcal{F}} f(F)$ over all fractional matchings $f$ .

Example: The Finite Projective Plane $\mathcal{P}_n$

$\nu(\mathcal{P}_n) = 1 \quad , \quad \nu^*(\mathcal{P}_n) = \frac{n^2+n+1}{n+1}$

Theorem [Aharoni-Berger-M]:

$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $\mathrm{I} \subset [m]$

$$\nu^*(\bigcup_{i \in \mathrm{I}} \mathcal{F}_i) > r(|\mathrm{I}| - 1) \quad .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.